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Massless phases for the Z_p model on the Union Jack lattice

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Abstract. We introduce and analyse Z_p-symmetric models on the Union Jack lattice. We show that these models have the same self-dual structure already known for other Z_p systems. By performing renormalisation group calculations as well as Monte Carlo simulations we analyse their phase diagrams, showing in particular that for $p \geqslant 5$ they exhibit a disordered massless phase.

1. Introduction

The XY model is one of the most interesting two-dimensional models. Despite the fact that it has a continuous global U(1) symmetry, which cannot be broken spontaneously in two dimensions (Mermin and Wagner 1966), it has a rather peculiar infinite-order phase transition (Kosterlitz and Thouless 1973). The low temperature phase is distinguished from the high temperature phase by its massless behaviour (infinite correlation length) and continuously varying critical exponents. Such a massless behaviour has been found (Elitzur *et al* 1979) even in the case where the symmetry is the discrete group Z_p (for $p \geqslant 5$), for the models whose $p \to \infty$ limit belongs to the same universality class as the XY model.

Recently a very large family of self-dual models involving multi-spin interactions was introduced (Alcaraz 1982), enlarging the class of models that exhibits massless phases. In particular, a somewhat different class of models with three-body interactions in the triangular lattice has been studied in detail (Alcaraz and Jacobs 1982a, b, Alcaraz *et al* 1983). The U(1) formulation of these triplet models shows an infinite-order phase transition from the massive paramagnetic phase to the disordered massless Gaussian phase, and the discrete version Z_p shows three phases for $p \geqslant p_c \simeq 5$, the intermediate phase being disordered and massless (Alcaraz and Jacobs 1982a, b, Alcaraz *et al* 1983).

In this paper we introduce and analyse a Z_p model involving triplet interactions on the Union Jack lattice. This model is the Z_p generalisation of the Ising model in the Union Jack lattice, which is known to be equivalent to the Baxter model (Hintermann and Merlini 1972). We show in § 2 that this model has the same self-dual structure as the preceding models, or more specifically, the Villain version (Villain 1975) of such models is self-dual for all p. We show in § 3 that the present model

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may be represented by a Coulomb gas of vector charges that interact with the logarithm of the intercharge distance. In § 4 by using this Coulomb gas representation the renormalisation group equations are derived for the U(1) model. The effect of a symmetry breaking field is analysed and other interesting related models are discussed in § 5. Finally in § 6 we present our results from Monte Carlo simulations, which agree very well with the earlier analyses exhibiting the three-phase picture for $p \geqslant 5$.

2. Definition and duality transformation of the model

2.1. Definition of the model

Consider the Union Jack lattice depicted in figure 1. This lattice is formed by three square sublattices, namely sublattice A with spacing a (θ sublattice) and sublattices B and C (ϕ sublattices) with spacing $\sqrt{2}a$ (see figure 1), toroidal periodic boundary conditions being assumed. We define Z_p variables at each point of the lattice:

$$S(\boldsymbol{r}) = \begin{cases} e^{i\theta(\boldsymbol{r})}, & \text{sublattice A,} \\ e^{i\phi(\boldsymbol{r})}, & \text{sublattices B and C,} \end{cases} \tag{2.1}$$

where $\theta(\boldsymbol{r}), \phi(\boldsymbol{r}) = 0, 2\pi/p, \ldots, 2(p-1)\pi/p$. Triangular variables may be defined by a product of variables around an elementary triangle involving a point of each sublattice

$$S_t \equiv S_i S_j S_k = \exp i(\phi_i + \theta_j + \phi_k) \equiv \exp i\phi_t. \tag{2.2}$$

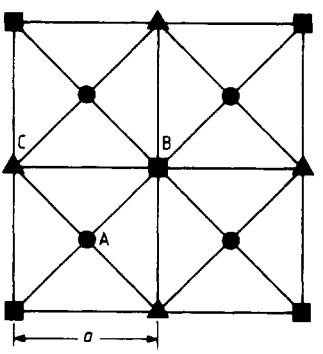

Figure 1. The Union Jack lattice. It is composed of the sublattice A (θ sublattice) denoted by circles and sublattices B and C (ϕ sublattices) denoted by squares and triangles respectively.

Let us now consider the general Z_p model (Cardy 1980, Alcaraz and Köberle 1980, 1981) defined by the reduced Hamiltonian

$$\mathcal{H} = -\sum_{\Delta} \left(\sum_{\eta=1}^{[p/2]} k_\eta (\cos \phi_t - 1) \right) \tag{2.3}$$

where the first sum is over all elementary triangles and $[p/2]$ is the integer part of $p/2$. The above model for $p = 2$ is equivalent to the Baxter model (Hintermann and Merlini 1972). The Potts version of the above model

$$\mathcal{H} = -\varepsilon \sum_{\Delta} \delta_k(\phi_t) \tag{2.4}$$

where δ_k is a Kronecker delta function, corresponds to the particular case of (2.3) in which $k_1 = k_2 = \cdots = \frac{1}{2}(1+(-)^p)k_{[p/2]} = \varepsilon$, and the clock (or vector Potts) version

$$\mathcal{H} = -k \sum_{\Delta} (\cos \phi_t - 1) \tag{2.5}$$

corresponds to the case $k_\alpha = k\delta_{\alpha,1}$.

The ground states of the above models have a p^2-fold degeneracy. These states are connected by the non-local transformation which rotates all the spins in one sublattice by an angle α $(=2\pi/p, 4\pi/p, \ldots, 2(p-1)\pi/p)$ and all the spins on another sublattice by an angle $2\pi - \alpha$. Such non-local symmetry may be broken spontaneously giving rise to a rich phase structure.

2.2. *Duality transformation*

We shall show in this section that the Villain form (Villain 1975) of the clock model (2.5) is self-dual for all p, therefore the self-dual structure for the general model (2.3) follows in a straightforward fashion.

The partition function for the Villain version of (2.5) is given by

$$Z_V = \sum_{\{J_t=-\infty\}}^{\infty} \sum_{\{\theta=0\}}^{2\pi(p-1)/p} \sum_{\{\phi=0\}}^{2\pi(p-1)/p} \exp\left(-\frac{k}{2}\sum_{\Delta}(\phi_t - 2\pi J_t)^2\right) \tag{2.6}$$

where the first sums are the trace over the θ, ϕ and J_t fields and the sum in the exponent is over all elementary triangles. The integer $\{J_t\}$ field, defined at each triangle, was introduced in order to keep the periodicity of the cosine form which is lost in a simple quadratic Gaussian approximation. By using the Poisson summation formulae we can write

$$\sum_{J_t(r)=-\infty}^{\infty} \exp\left(-\frac{k}{2}(\phi_t - 2\pi J_t)^2\right) = \frac{1}{(2\pi k)^{1/2}} \sum_{l_t=-\infty}^{\infty} \exp\left(-\frac{l_t^2}{2k} + \mathrm{i} l_t \phi_t\right) \tag{2.7}$$

where l_t is an integer field defined in the triangles. By writing the field l_t in terms of two other fields $l_t = p\rho_t + \nu_t$ with $-\infty < \rho_t < \infty$, $0 \leq \nu_t \leq p-1$ the partition function takes the form

$$Z_V = \sum_{\{\rho_t=-\infty\}}^{\infty} \sum_{\{\nu_t=0\}}^{p-1} \exp\left(-\frac{1}{2k}\sum_{\Delta}(p\rho_t + \nu_t)^2\right) \sum_{\{\theta,\phi=0\}}^{2\pi(p-1)/p} \exp \mathrm{i} \sum_{\Delta} \phi_t \nu_t \tag{2.8}$$

where we have dropped a harmless constant. To proceed we must perform the θ and ϕ summations; in order to do this we must isolate (integrate by parts) the θ and ϕ variables, by writing

$$\sum_{\Delta} \phi_t \nu_t = \sum_{r \in A} \theta(\boldsymbol{r})\nu^*(\boldsymbol{r}) + \sum_{r \in B,C} \phi(\boldsymbol{r})\nu^*(\boldsymbol{r}) \tag{2.9}$$

where $\nu^*(\boldsymbol{r})$ (see figure 2) is defined as the sum (mod p) of ν_t variables attached to the triangles which have the point $\boldsymbol{r}$ as a corner,

$$\nu^*(\boldsymbol{r}) = \begin{cases} \nu_{t_1} + \nu_{t_2} + \cdots + \nu_{t_8} \pmod p, & \phi\text{-sublattice}, \\ \nu_{t_1} + \nu_{t_2} + \nu_{t_9} + \nu_{t_{10}} \pmod p, & \theta\text{-sublattice}. \end{cases} \tag{2.10}$$

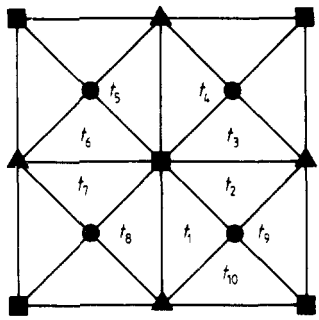

Figure 2. A point in the ϕ sublattice is surrounded by eight triangles while a point in the θ sublattice is surrounded by four.

By inserting (2.9) in (2.8) and performing the θ–ϕ summations we obtain

$$Z_V = \sum_{\{\rho_t=-\infty\}}^{\infty} \sum_{\{\nu_t=0\}}^{p-1} \exp -\frac{1}{2k} \sum_{\Delta} (p\rho_t + \nu_t)^2 \prod_{r} p\delta_k[\nu^*(\boldsymbol{r})] \tag{2.11}$$

where δ_k is a Kronecker delta since $\nu^*(\boldsymbol{r})$ is defined modulo p. We must now define dual fields so that the δ constraints

$$\nu^*(\boldsymbol{r}) = 0 \ \forall \boldsymbol{r}, \qquad 0 \leq \nu_t \leq p-1 \ \forall t, \tag{2.12}$$

are satisfied. The dual variables $\tilde{\theta}(\boldsymbol{r})$, $\tilde{\phi}(\boldsymbol{r})$ $(\tilde{\theta}(\boldsymbol{r},)\tilde{\phi}(\boldsymbol{r}) = 0, 2\pi/p, \ldots, 2\pi(p-1)/p)$ are defined on the same lattice subject to periodic boundary conditions.

Triangular dual variables are defined in the same way as (2.2),

$$\tilde{\phi}_t = \tilde{\phi}_1 + \tilde{\theta}_2 + \tilde{\phi}_3. \tag{2.13}$$

It is easy to verify that in terms of the dual variables the solution of (2.12) is given by

$$\nu_t = \varepsilon_t (p/2\pi)\tilde{\phi}_t \ (\mathrm{mod}\, p) \qquad \varepsilon_t^2 = 1, \tag{2.14}$$

where the sign of ε_t alternates from one triangle to the other. Conversely given a ν_t configuration that satisfies (2.12) we can find a dual $\tilde{\theta}$, $\tilde{\phi}$ configuration:

$$\tilde{\theta}(x, y) = \frac{2\pi}{p} \varepsilon_t(x, y - \tfrac{1}{4}) \sum_{n=0}^{\infty} \nu_t(x, y - \tfrac{1}{4}n - \tfrac{1}{4}), \tag{2.15a}$$

$$\tilde{\phi}(x, y) = \frac{2\pi}{p} \varepsilon_t(x + \tfrac{1}{2}, y - \tfrac{1}{4}) \times \sum_{n=0}^{\infty} (\nu_t(x + n + \tfrac{1}{2}, y - n - \tfrac{1}{4}) + \nu_t(x + n + \tfrac{3}{4}, y - n - \tfrac{1}{2})) \tag{2.15b}$$

in which $\nu_t(x, y)$ is the variable attached to the triangle whose centre is located at (x, y). It must be clear that the infinite string in equations (2.15) may be modified by the addition of any $\nu^*(\boldsymbol{r})$. The correspondence between ν_t variables and the dual variables $\tilde{\theta}$, $\tilde{\phi}$ is not one-to-one, because there are many $\tilde{\theta}$, $\tilde{\phi}$ configurations (related by the non-local symmetries discussed earlier) that produce the same ν_t configuration. In replacing the ν_t summation by $\tilde{\theta}$, $\tilde{\phi}$ summations we will therefore commit an

overcounting, but it is not difficult to convince oneself (Savit 1980) that this overcounting is uniform. Thus, apart from a harmless constant, (2.11) may be written as

$$Z_V = \sum_{\{\rho_t=-\infty\}}^{\infty} \sum_{\{\theta,\phi=0\}}^{2\pi(p-1)/p} \exp -\frac{p^2}{2k(2\pi)^2} \sum_{\Delta} (\tilde{\phi}_t - 2\pi\rho_t)^2 \tag{2.16}$$

in which the transformation $\rho_t \to -\rho_t\varepsilon_t$ has been made. By comparison with equation (2.6) we have then obtained that (apart from an unimportant constant)

$$Z_V(k) = Z_V(\tilde{k}) \tag{2.17}$$

which states that the Villain model is self-dual for all p, the dual inverse temperature given by $\tilde{k} = p^2/(2\pi)^2 k$.

We can in a straightforward fashion perform the duality transformation for the general model defined in (2.3). We obtain the same self-dual structure already known for the other Z_p systems mentioned in the introduction of this paper. For all p the Potts version defined by equation (2.4) is self-dual, and the clock version (2.5) is self-dual for $p < 5$.

To conclude this section it is useful to consider the case in which the symmetry is continuous ($Z_{p \underset{p\to\infty}{\to}} U_1$). The θ and ϕ variables become continuous angles $0 \leqslant \theta, \phi \leqslant 2\pi$ and the partition function in the Villain form is given by

$$Z_V = \int_0^{2\pi} D\theta D\phi \sum_{\{J_t=-\infty\}}^{\infty} \exp -\frac{k}{2} \sum_{\Delta} (\phi_t - 2\pi J_t)^2 \tag{2.18}$$

where the integrations are the trace of the θ, ϕ angles and ϕ_t is defined in the same way as (2.2). By using the Poisson formulae (see equation (2.7)) and isolating the θ and ϕ angles as in equation (2.9), we can perform the θ–ϕ integrals, obtaining apart from a constant term

$$Z_V = \sum_{\{\nu_t=-\infty\}}^{\infty} \exp -\frac{1}{2k} \sum_{\Delta} \nu_t^2 \prod_{r} \delta_k[\nu^*(r)] \tag{2.19}$$

where $\nu^*(r)$ is defined by equation (2.9). We now introduce dual integer variables $-\infty < \tilde{\theta}(r), \tilde{\phi}(r) < \infty$ at each point of the lattice so that the δ_k requirements of equation (2.19) are satisfied by setting

$$\nu_t = \varepsilon_t \nu \tilde{\phi}_t = \varepsilon_t(\tilde{\theta}_i + \tilde{\phi}_j + \tilde{\phi}_k), \qquad \varepsilon_t^2 = 1, \tag{2.20}$$

where $\tilde{\phi}_t$ are the triangular dual fields defined by (2.13) and ε_t alternate in sign from one triangle to another. By inserting equation (2.20) in (2.19) we obtain the dual form

$$Z_V = \sum_{\{\tilde{\theta}(r)=-\infty\}}^{\infty} \sum_{\{\tilde{\phi}(r)=-\infty\}}^{\infty} \exp -\frac{1}{2k} \sum_{\Delta} (\tilde{\theta}_i + \tilde{\phi}_j + \tilde{\phi}_k)^2. \tag{2.21}$$

This form will be useful to obtain a charge gas representation for the model.

3. Coulomb gas representation

We consider here the model with continuous symmetry. For the case in which the symmetry is discrete we obtain in a straightforward fashion a double gas representation in the same way as the two-dimensional Z_p model in the square lattice (Kadanoff

1978, Elitzur *et al* 1979). The results of the previous section show that after duality the partition function can be written

$$Z = \operatorname*{Tr}_{\tilde\theta,\tilde\phi,m_\theta,m_\phi} \exp\Big(-\frac{1}{2k}\sum_{\Delta}(\tilde\theta_i+\tilde\phi_j+\tilde\phi_k)^2 + 2\pi i\sum m_\theta(\boldsymbol{r})\tilde\theta(\boldsymbol{r}) + 2\pi i\sum m_\phi(\boldsymbol{r})\tilde\phi(\boldsymbol{r})\Big) \tag{3.1}$$

where we have used the Poisson sum formula. The integers m_θ, m_ϕ are respectively defined on the sites where the original angles θ, ϕ were defined. The result of the Gaussian integration over $\tilde\theta$, $\tilde\phi$ can be written in matrix form,

$$Z = \operatorname*{Tr}_{m_\theta,m_\phi} \exp\Big[-2\pi^2 k\sum_{\boldsymbol{r},\boldsymbol{r}'}(m_\theta(\boldsymbol{r})m_\phi(\boldsymbol{r}))\boldsymbol{G}(\boldsymbol{r}-\boldsymbol{r}')\begin{pmatrix}m_\theta(\boldsymbol{r}')\\ m_\phi(\boldsymbol{r}')\end{pmatrix}\Big] \tag{3.2}$$

where

$$\boldsymbol{G}(\boldsymbol{r}-\boldsymbol{r}') = \frac{a^2}{8}\int\frac{d^2k}{(2\pi)^2}\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{c_1^2+c_2^2-2c_1^2c_2^2}\begin{pmatrix}2(c_1^2+c_2^2) & -2c_1c_2\\ -2c_1c_2 & 1\end{pmatrix} \tag{3.3}$$

with $c_i = \cos(k_i a/2)$. The integration is over the first Brillouin zone $|k_i| < \pi/a$. Equation (3.3) is not well defined, however, since the integrand has singularities in the infinite volume limit.

There are two such singularities which are not equivalent by a reciprocal lattice vector: at $\boldsymbol{k} = 0$ ($c_1 = c_2 = 1$), and at $\boldsymbol{k} = (\pi/a, \pi/a)$ ($c_1 = c_2 = 0$). The latter singularity occurs only in the (ϕ, ϕ) matrix element.

In order to have quantities which are finite in the thermodynamic limit, we may consider

$$G'_{\theta\theta}(\boldsymbol{r}) \equiv G_{\theta\theta}(\boldsymbol{r}) - G_{\theta\theta}(0) = \frac{a^2}{4}\int\frac{d^2k}{(2\pi)^2}\frac{(c_1^2+c_2^2)(e^{i\boldsymbol{k}\cdot\boldsymbol{r}}-1)}{c_1^2+c_2^2-2c_1^2c_2^2} \tag{3.4}$$

$$G'_{\theta\phi}(\boldsymbol{r}) \equiv G_{\theta\phi}(\boldsymbol{r}) + \tfrac{1}{2}G_{\theta\theta}(0) = \frac{a^2}{8}\int\frac{d^2k}{(2\pi)^2}\frac{(-2c_1c_2)\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}+c_1^2+c_2^2}{c_1^2+c_2^2-2c_1^2c_2^2} \tag{3.5}$$

$$G'_{\phi\phi}(\boldsymbol{r}) \equiv G_{\phi\phi}(\boldsymbol{r}) - \Delta(\boldsymbol{r},0)G_{\phi\phi}(0) = \Delta(\boldsymbol{r},0)\frac{a^2}{8}\int\frac{d^2k}{(2\pi)^2}\frac{(e^{i\boldsymbol{k}\cdot\boldsymbol{r}}-1)}{c_1^2+c_2^2-2c_1^2c_2^2}. \tag{3.6}$$

In (3.6), $\Delta(\boldsymbol{r}, \boldsymbol{r}')$ is defined to be $+1$ if $\boldsymbol{r}$, $\boldsymbol{r}'$ are on the same sublattice, and zero otherwise. This is to take account of the fact that $G_{\phi\phi}(\boldsymbol{r}-\boldsymbol{r}')$ vanishes when $\boldsymbol{r}$, $\boldsymbol{r}'$ are on different ϕ sublattices. Note also that the substitution $k_i \to k_i + (\pi/2a)$ shows that $G_{\phi\phi}(0) = \frac{1}{2}G_{\theta\theta}(0)$, even in the finite volume system.

Isolating the terms in (3.2) proportional to $G_{\theta\theta}(0)$, we have

$$G_{\theta\theta}(0)\sum_{\boldsymbol{r},\boldsymbol{r}'}[m_\theta(\boldsymbol{r})m_\theta(\boldsymbol{r}') + \tfrac{1}{2}m_\phi(\boldsymbol{r})m_\phi(\boldsymbol{r}')\Delta(\boldsymbol{r},\boldsymbol{r}') - m_\theta(\boldsymbol{r})m_\phi(\boldsymbol{r}')] \tag{3.7}$$

which may be written

$$G_{\theta\theta}(0)\Big|\sum_{\boldsymbol{r}}[m_\theta(\boldsymbol{r}) - \tfrac{1}{2}(1+i\varepsilon(\boldsymbol{r}))m_\phi(\boldsymbol{r})]\Big|^2 \tag{3.8}$$

where $\varepsilon(\boldsymbol{r}) = \pm 1$ according to which sublattice $\boldsymbol{r}$ lies on. This expression suggests that we define a complex, or vector-valued, vortex charge

$$m(\boldsymbol{r}) = m_\theta(\boldsymbol{r}) - \tfrac{1}{2}(1+i\varepsilon(\boldsymbol{r}))m_\phi(\boldsymbol{r}). \tag{3.9}$$

In the infinite volume limit, as $G_{\theta\theta}(0) \to +\infty$, the configurations with finite energy must satisfy the charge neutrality condition $\Sigma_r m(r) = 0$, or, in terms of real quantities,

$$\sum_r (m_\theta(r) - \tfrac{1}{2} m_\phi(r)) = 0, \qquad \sum_r \varepsilon(r) m_\phi(r) = 0. \tag{3.10a, b}$$

The fundamental vector charges are illustrated in figure 3(*a*). Similar charge lattices arise in the theory of anisotropic two-dimensional melting (Ostlund and Halperin 1981).

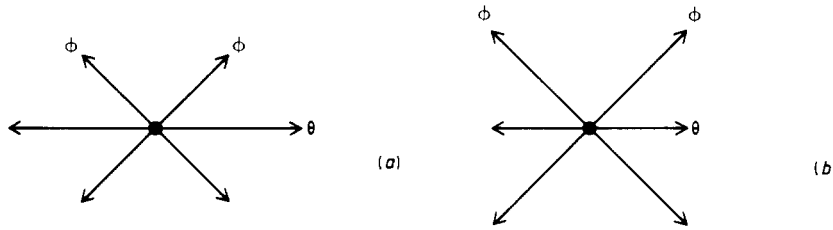

Figure 3. Fundamental vector charges. (*a*) corresponds to the charges (3.9) and (*b*) to the charges (5.4).

The asymptotic behaviour of the $G'(r)$ as $r \to \infty$ is evaluated in the appendix. There we find

$$G'_{\theta\theta}(r) \sim -(1/\pi) \ln(|r/a|) - 2C, \tag{3.11a}$$

$$G'_{\theta\phi}(r) \sim (1/2\pi) \ln(|r/a|) + C, \tag{3.11b}$$

$$G'_{\phi\phi}(r) \sim -(1/2\pi)\Delta(r, 0) \ln(|r/a|) - \Delta(r, 0)C, \tag{3.11c}$$

where we have dropped terms of $O(a/r)^2$ which are irrelevant in the RG analysis. The value of C is estimated to be 0.19.

With the approximations (3.11), the Coulomb gas Hamiltonian takes the simple form

$$2\pi k \sum_{r \neq r'} m(r)^* m(r') \ln\left(\left|\frac{r - r'}{a}\right|\right) + \ln y_\theta \sum_r m_\theta(r)^2 + \ln y_\phi \sum_r m_\phi(r)^2 \tag{3.12}$$

where the initial values of the vortex fugacities are given by

$$\ln y_\theta = -4\pi^2 kC, \qquad \ln y_\phi = -2\pi^2 kC. \tag{3.13a, b}$$

4. Renormalisation group analysis

Following Kosterlitz (1974), we consider an expansion of the Coulomb gas partition function in the vortex fugacities y_θ, y_ϕ, making the approximations (which should not affect universal quantities) of treating the lattice as a continuum, with a hard core repulsion between the vortices. In this case, the radius of this hard core depends on the nature of the vortices. For two θ vortices it is a, for two ϕ vortices it is $\sqrt{2}a$ (ϕ vortices on different sublattices do not interact), while for a θ–ϕ pair it is $a/\sqrt{2}$.

To lowest order, the partition function is

$$Z = 1 + y_\theta^2 \int_{|r_1 - r_2| > a} \frac{d^2 r_1 \, d^2 r_2}{a^4} \left| \frac{r_1 - r_2}{a} \right|^{-4\pi k} + \tfrac{1}{2} y_\phi^2 \int_{|r_1 - r_2| > \sqrt{2}a} \frac{d^2 r_1 d^2 r_2}{a^4} \left| \frac{r_1 - r_2}{a} \right|^{-2\pi k} + \cdots . \tag{4.1}$$

The change $a \to a(1+\delta l)$ ($\delta l \ll 1$) may be compensated by an appropriate change in y_θ, y_ϕ. This leads to the RG equations

$$dy_\phi / dl = (2 - \pi k) y_\phi + A y_\phi y_\theta, \tag{4.2a}$$

$$dy_\theta / dl = (2 - 2\pi k) y_\theta + B y_\phi^2, \qquad dk/dl = D y_\phi^2 + E y_\theta^2. \tag{4.2b, c}$$

The $O(y_\phi y_\theta)$ term in (4.2a) arises because there are some configurations where a θ vortex and a ϕ vortex approach within a distance $a(1+\delta l)/\sqrt{2}$ of each other. Such configurations will be counted as ϕ vortices in the renormalised partition function (vectorially, a θ vortex plus a ϕ vortex can make a ϕ vortex). This will come from a region of space of area $2\pi(a/\sqrt{2})^2 \delta l$, and the appropriate Boltzmann weight is $\exp[-2\pi k \ln(1/\sqrt{2})]$. Thus $A = \pi \exp(\pi k \ln 2)$. Similarly, a configuration of two ϕ vortices (on different sublattices) generates a θ vortex under renormalisation, so that $B = \pi/2$.

The renormalisation of k, (4.2c), arises because neutral pairs with separation $< a(1+\delta l)$ screen the interaction between other charges. The magnitude of these terms may be inferred from the Kosterlitz (1974) result, taking into account the different core radii and interaction strengths. Thus we find that $D = -4\pi^3 k^2 \exp(-\pi k \ln 2)$, $E = -8\pi^3$.

The RG equations (4.2) are valid only to $O(y^2)$, but this is sufficient to discuss the critical behaviour. For sufficiently large k we renormalise onto the Gaussian model $y_\theta = y_\phi = 0$. As k decreases, eventually y becomes relevant and the ϕ vortices unbind. The second term in (4.2b) will force the θ vortices to unbind at the same time. This is because a θ vortex may be viewed as a bound state of two ϕ vortices, so that the self-interaction of the θ vortices is screened. Close to the critical fixed point $\pi k = 2$ we see from (4.2b) that y_θ goes exponentially towards $\frac{1}{4}\pi y_\phi^2$ so the y_θ terms in (4.2a, c) may be neglected. (Note that initially $y = y_\phi^2$.) The resulting equations are just those of the usual XY model.

The critical separatrix has the form

$$y_\phi = 1/\pi - 2/k\pi^2 \tag{4.3}$$

in the linear approximation. The critical temperature $T_c = k_c^{-1}$ for the Villain U(1) model is estimated by the intersection of the fugacity given by (3.13b) and the critical separatrix (4.3). This gives us

$$T_c \sim 1.30. \tag{4.4}$$

The asymptotic behaviour of the correlation functions for $T < T_c$ may be computed within the Gaussian model. We find

$$c_{\theta\theta}(r - r') \equiv \langle e^{i\theta(r)} e^{-i\theta(r')} \rangle \sim |r - r'|^{-2\eta}, \tag{4.5a}$$

$$c_{\phi\phi}(r - r') \equiv \langle e^{i\phi(r)} e^{-i\phi(r')} \rangle \sim \Delta(r, r') |r - r'|^{-\eta}, \tag{4.5b}$$

where $\eta = 1/2\pi k(\infty)$, $k(\infty)$ being the renormalised value of k at infinite length scales. At T_c, $\eta = \frac{1}{4}$ as for the usual XY model.

The estimated temperature (4.4) is quite useful to determine an upper bound on the critical p_c, such that the discrete Z_p model with $p > p_c$ must have three phases. Following Elitzur *et al* (1979) and Alcaraz *et al* (1983), we can obtain in the Villain formulation rigorous inequalities which state that correlation functions of ordered (disordered) variables are stronger (weaker) in the Z_p model than in the U(1) model. Using these inequalities, and the fact that the Villain model is self-dual, standard arguments (Elitzur *et al* 1979, Alcaraz *et al* 1983) imply that for $p > 2\pi/T_c$ (where T_c is the critical temperature for the U(1) Villain model) the discrete Villain Z_p model should have three phases, the intermediate one being disordered and massless. By using the estimate (4.4) we obtain $2\pi/T_c = 4.8$. Because the Boltzmann weights of the clock and Villain model are almost identical, we believe that also for the clock model for $p \geq 5$ we should have three phases, which agrees very well with our Monte Carlo simulations presented in § 6.

5. Symmetry breaking field perturbations

Another interesting point to be studied by using renormalisation group ideas is the effect of symmetry breaking fields in the scenario hitherto presented. Those fields are introduced by adding to the Hamiltonian a term

$$h_p^{\theta} \sum_{r \in \theta} \cos p\theta + h_p^{\phi} \sum_{r \in \phi} \cos p\phi. \tag{5.1}$$

However, an equivalent but more useful way to introduce this effect (José *et al* 1977) is to write the partition function

$$Z = \operatorname*{Tr}_{\theta,\phi,n_\theta,n_\phi} \exp\Big(k \sum_{\Delta} v(\phi_i + \theta_j + \phi_k) + ip \sum_{r \in \theta} n_\theta(r)\theta(r) + ip \sum_{r \in \phi} n_\phi(r)\phi(r) + \ln y_p^{\theta} \sum_{r \in \theta} n_\theta^2(r) + \ln y_p^{\phi} \sum_{r \in \phi} n_\phi^2(r) \Big) \tag{5.2}$$

where θ, ϕ are continuous variables and the integer fields n_θ and n_ϕ are defined on the θ and ϕ sublattices respectively. The parameters y_p^{θ} and y_p^{ϕ} play the role of symmetry breaking fields; in fact, as y_p^{θ}, $y_p^{\phi} \to 0$ one obtains the U(1) model with a small symmetry breaking field, but, on the other hand, in the limit y_p^{θ}, $y_p^{\phi} \to 1$, the summation over the integers n_θ, n_ϕ forces the angles to be multiples of $2\pi/p$, and therefore one obtains the Z_p model.

In the low temperature limit, vortices are irrelevant, and v may be replaced by a Gaussian. The Gaussian integration over θ and ϕ then leads to a Coulomb gas with the selfsame form as (3.12), with the replacements $k \to p^2/4\pi^2 k$, $m_\theta \to n_\theta$, $m_\phi \to n_\phi$. The discrete excitations should therefore be irrelevant for renormalised values of $k^{-1} > 8\pi/p^2$. Consequently there will be a portion of the Gaussian line for which both vortices and discrete excitations are irrelevant if $p > p_c = 4$. Our Monte Carlo simulations (see § 6) show that the discrete model (y_p^{θ}, $y_p^{\phi} \to 1$) has a massless phase for $p > 4$, and hence the strip of fixed points extends from the Gaussian model (y_p^{θ}, $y_p^{\phi} = 0$) to the discrete model for $p > 4$ as sketched in figure 4(a).

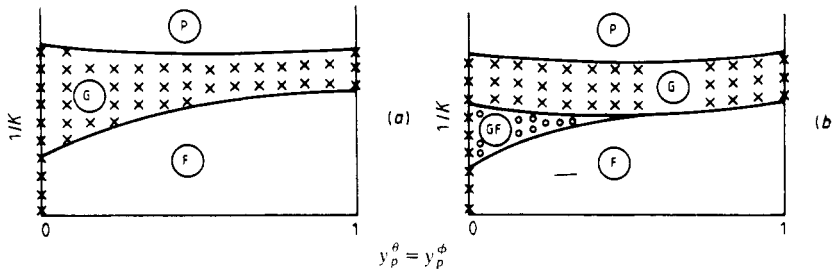

Figure 4. Phase diagrams. The phases denoted by F, P and G are the ferromagnetic, paramagnetic and massless Gaussian respectively. The points represented by × are points with power law decay for the θ and ϕ variables. (*a*) corresponds to the model $v(q\theta+\phi+\phi)$ with $p \geq 5$, $q \leq 2$ and (*b*) to the model $v(q\theta+\phi+\phi)$ with $q \geq 3$ and $p > 2\sqrt{2}q$. The phase GF that appears in (*b*) is ordered in the θ variables but has a power law decay in the ϕ variables.

The same type of analysis may be applied to models with an interaction of the form

$$v(q\theta+\phi+\phi) \tag{5.3}$$

where q is a positive integer. Following through the analysis of the previous section, we see that the vortex gas is unchanged by this modification, while by a change of variables $q\theta \to \theta$, (5.1) is modified by the change $n_\theta \to q^{-1}n_\theta$. We end up with a Coulomb gas of the form (3.12), but with the complex charges

$$n(\boldsymbol{r}) = q^{-1}n_\theta(\boldsymbol{r}) - \tfrac{1}{2}(1+\mathrm{i}\varepsilon(\boldsymbol{r}))n_\phi(\boldsymbol{r}). \tag{5.4}$$

The fundamental vector charges for the case $q=2$ are shown in figure 3(*b*). Note that for $q \geq 2$, the θ charges couple more weakly than the ϕ charges and thus unbind first, at a renormalised $k^{-1} = 4\pi q^2/p^2$. Since the vortices still unbind when $k^{-1} = \pi/2$, we find that there is a phase in which the θ variables are ordered and the ϕ variables disordered for small symmetry breaking field perturbations with $p > p_c = 2\sqrt{2}q$. When the θ charges unbind, they are able to screen only the real part of the ϕ charges. The imaginary parts of the ϕ charges still interact logarithmically, with a charge reduced by a factor $1/\sqrt{2}$. Therefore, there is in principle a second transition when the ϕ charges unbind, at $k^{-1} = 16\pi/p^2$. This will be distinct from the θ transition for $q \geq 3$. Thus the full model with $q > 2$ and small symmetry breaking field perturbations with $p > 2\sqrt{2}q$ should exhibit three transitions in all. In the low temperature phase both $\langle \mathrm{e}^{\mathrm{i}\theta}\rangle$ and $\langle \mathrm{e}^{\mathrm{i}\phi}\rangle$ will be non-zero; in the next phase only $\langle \mathrm{e}^{\mathrm{i}\theta}\rangle$ will order, and the correlation function $\langle \mathrm{e}^{\mathrm{i}\phi(\boldsymbol{r})}\,\mathrm{e}^{-\mathrm{i}\phi(\boldsymbol{r}')}\rangle$ will have power law decay, followed by a completely massless phase, then a high temperature massive phase.

One question that naturally arises is whether these conclusions based on small breaking field perturbations in the Gaussian model still persist for the truly discrete model (h_p^θ, $h_p^\phi \to \infty$ or y_p^θ, $y_p^\phi \to 1$). To consider this question we pay attention to the case in which q/p and p/q are not integers. In this case it is easy to convince oneself that the partition function for the discrete Z_p model $v(q\theta+\phi+\phi)$ is the same as for the discrete Z_p model with $q=1$. Hence, as our numerical calculations presented in the next section indicate, we expect only two phase transitions for the truly discrete model. Therefore we believe that at sufficiently large breaking field strength two of these phases (that appear for small breaking fields) collapse, giving rise to the phase diagram sketched in figure 4(*b*).

6. Numerical analysis

We have performed Monte Carlo simulations for the clock models defined in equation (2.5). We have employed the Metropolis updating algorithm (Metropolis *et al* 1953) and a variant of the heatbath algorithm of Creutz *et al* (1979a, b) whenever the large number of degenerate states render the first method inefficient.

Thermal loops were obtained to give a general overview of the phase structure of the model. We initialise the system in an ordered state at a high inverse temperature $K = K_0$, and the system is heated in small steps ΔK until $K = 0$ is reached, where we invert the procedure by cooling the system until $K = K_0$.

A single Monte Carlo iteration of the entire lattice is performed for each step in K, and averages of interest are measured. It is clear that the system (with only a single iteration) will not have 'time' to reach an equilibrium state in such a simulation. However if the the variation in K is small enough the system will be close to equilibrium except near a phase transition in which the relaxation time increases and the system becomes far from equilibrium. Therefore the thermal loops will show hysteresis near a critical point.

In figures 5(*a*)–(*e*) we show some examples of thermal loops for the case Z_2, Z_3, Z_4, Z_5 and Z_7 of the Hamiltonian (2.5) for a lattice of 1600 points. We have measured in each Monte Carlo step the internal energy per triangle

$$E = \frac{1}{N_\Delta} \sum_\Delta (1 - \cos \phi_t)$$

where N_Δ is the number of triangles of the lattice. We see clearly that the Z_2, Z_3 and Z_4 models show only one phase transition while Z_5 and Z_7 show two phase transitions. We must stress that because of the large degeneracy of the Hamiltonian in each step we accept or reject a chosen spin by testing the entire group before going to the next spin.

We can obtain a quite accurate value for the critical temperature by observing the time evolution of a mixed state in which half the system has been initialised with random values and the rest ordered. For details of this technique we refer to Creutz *et al* (1979a, b), Alcaraz and Jacobs (1982b). Our numerical results give the self-dual temperatures for $p \leqslant 4$ to an accuracy of ± 0.005, that is $k_2^c = \frac{1}{2}\ln(1+\sqrt{2}) \simeq 0.441$, $k_3^c = \frac{2}{3}\ln(\sqrt{3}+1) \simeq 0.670$, $k_4^c = 0.881$. For $p \geqslant 5$ the first transition practically remains fixed around $k_I = 1$ while the second one scales with the inverse gap, that is, $k_{II} \simeq \gamma/(1-\cos 2\pi/p)$ with $\gamma \simeq 0.86$.

In order to determine the order of these transitions we observe, at the critical temperature (Creutz *et al* 1979a, b, Alcaraz and Jacobs 1982b), the time evolution of an initially ordered state and an initially disordered state. In figures 6(*a*), (*b*) we show such time evolution for the Z_3 and Z_4 systems respectively, which indicates the existence of latent heat which therefore renders the transition to be first order. For $p = 2$ as well as for $p \geqslant 5$ the time evolution does not show such gaps, which indicates that the transitions are continuous (no latent heat).

To verify the massless behaviour of the intermediate phase we measured the magnetic susceptibility per spin (Alcaraz and Jacobs 1982b), for points in such a phase for several lattice sizes. We observe that this quantity grows with the lattice size and does not saturate, which is consistent with the correlation length in such a phase being infinite for the infinite system. In figure 7 we show that the pure spin wave behaviour fits well with the behaviour in the intermediate phase for the Z_{19} model.

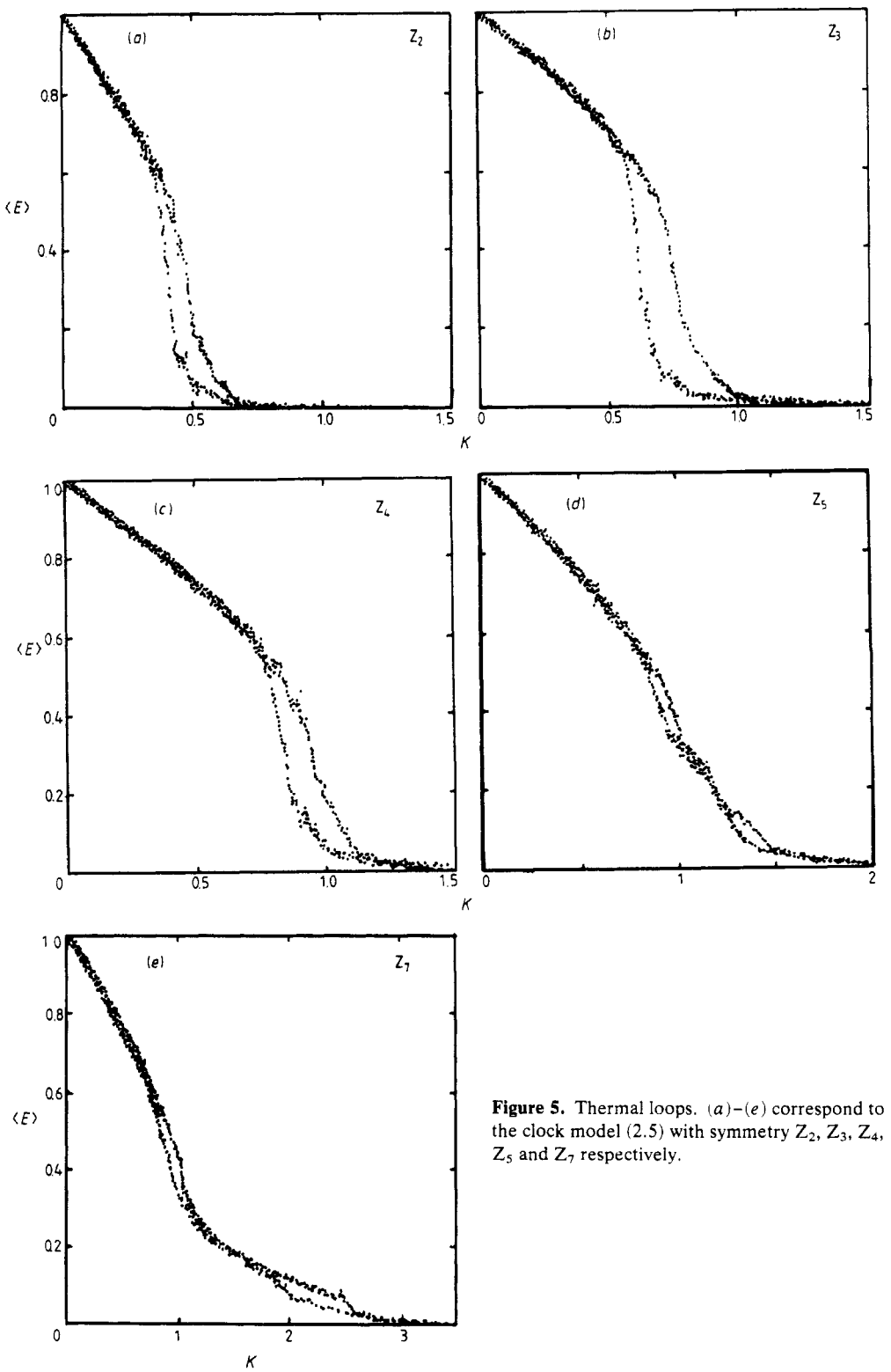

Figure 5. Thermal loops. (*a*)–(*e*) correspond to the clock model (2.5) with symmetry Z_2, Z_3, Z_4, Z_5 and Z_7 respectively.

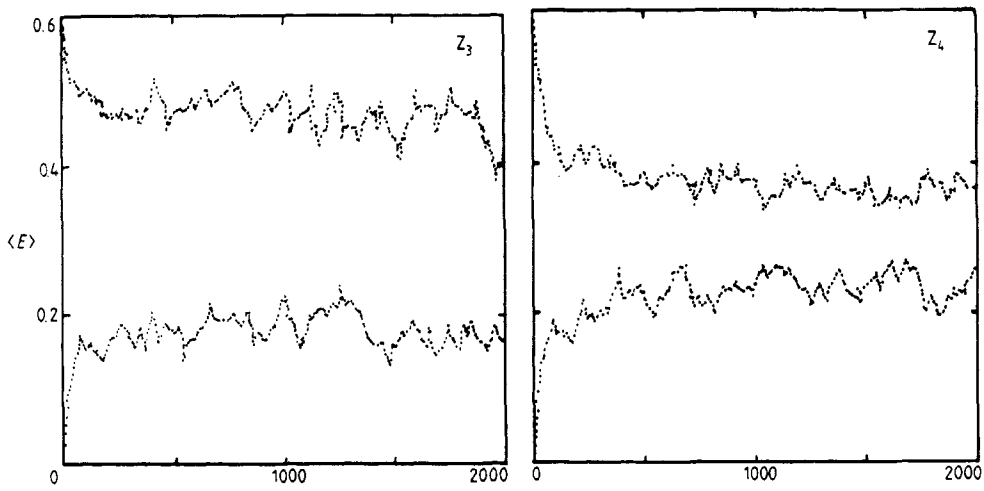

Figure 6. (*a*) Time evolution at the critical temperature of an initially ordered state (lower points) and an initially disordered state (upper points) for the Z_3 model with 3600 spins. The unit of time is represented by one Monte Carlo iteration. The result shows two apparently stable states at the critical temperature indicating a first-order transition. (*b*) Same as (*a*) for the Z_4 model with 4900 spins.

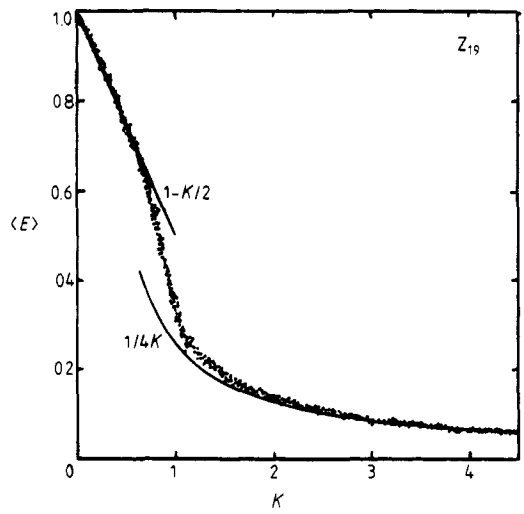

Figure 7. Heating branch of a thermal cycle for the Z_{19} model. The continuous straight (curved) lines represent the first term in a high temperature expansion (spin wave approximation).

The same numerical analysis performed hitherto for the clock model (2.5) has been done for the clock version of the model (5.3) with $q = 2$. We have obtained, as in the case $q = 1$, three phases for $p \geqslant 5$, which agrees with our earlier renormalisation group analysis.

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Appendix

We compute the asymptotic behaviour of the Green functions G'. Consider $G'_{\theta\theta}(\boldsymbol{r})$; the leading behaviour as $r \to \infty$ is given from the vicinity of the singularity at $\boldsymbol{k} = 0$. Expanding the integrand in the neighbourhood of this singularity,

$$G'_{\theta\theta}(\boldsymbol{r}) \sim \frac{a^2}{4} \int^{a^{-1}} \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{8}{k^2} (\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} - 1) \sim -\frac{1}{\pi} \ln(|\boldsymbol{r}/a|) - 2C. \tag{A1}$$

The others are evaluated similarly. To see that the constant term is universal, consider for example

$$G'_{\theta\theta}(\boldsymbol{r}) + 2G'_{\theta\phi}(\boldsymbol{r}) = \frac{a^2}{4} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{(c_1^2 + c_2^2 - 2c_1 c_2)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}}{c_1^2 + c_2^2 - 2c_1^2 c_2^2}. \tag{A2}$$

It is easy to see by power counting that this integral is $O(a/r)^2$ as $r \to \infty$. To estimate C, we may use the fact that, as may be shown by direct integration, $G'_{\phi\phi}(\sqrt{2}a) = -\frac{1}{4}$, and assume the form (3.11c) to be correct down to this separation. This gives $C \approx 0.19$. This may be checked by observing that $G'_{\theta\phi}(a/\sqrt{2}) = \frac{1}{8}$, which leads to $C \approx 0.18$. At the level of accuracy we need for the RG equations, this error is satisfactory.

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